Communication for maths

On the formal presentation of differentiation – part 4: Writing mathematics.

Introduction

- We have studied some differentiation, and we will soon study integration.
- The communication of differentiation and integration can become very technical, both in its mathematics and its mathematical English.

Introduction

- The theoretical study of the topics of limits, functions, infinite processes and calculus is called *real analysis*.

- The communication of real analysis (i.e. the mathematics and the mathematical English) is highly technical.

Introduction

Example 1

1)

Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function bounded above and below.

Then f is continuous and attains a maximum or minimum.

2) Every differentiable bounded real function is continuous

and attains an extrema.

Introduction

Example 2

1) Let $f: \mathbb{R} \to \mathbb{R}$ be a function, and let $x_0 \in \mathbb{R}$.

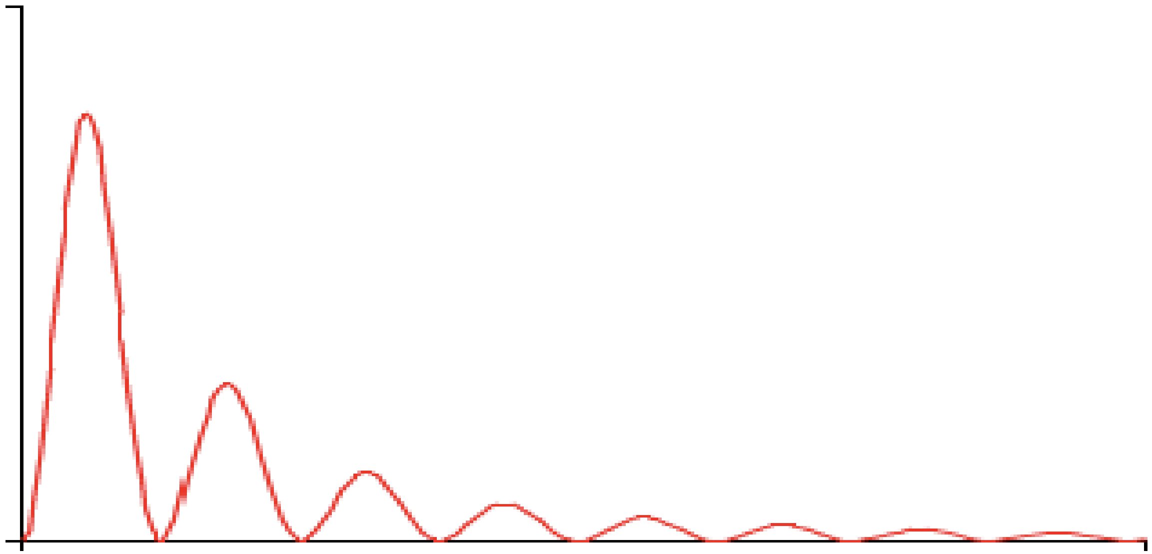
   If f is differentiable and increasing at x_0 then $f'(x_0) \geq 0$.

2) Every function which is increasing and differentiable at a point has a positive gradient at that point.

Introduction

Example 3:

A physics based description to the following:



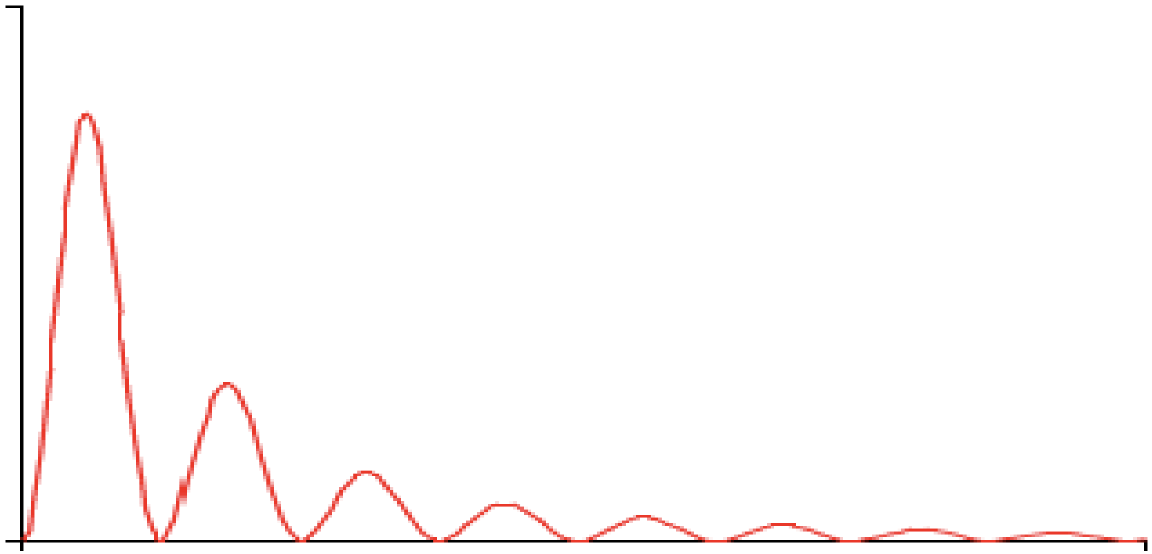
This function displays regular oscillations of constant period, with amplitude decreasing to zero.

From p87, “Mathematical Writing for undergraduate students”, Franco Vivaldi, Queen Mary, University of London

Introduction

Exercise 1

- Give a mathematical description to the function on the right



From p87, “Mathematical Writing for undergraduate students”, Franco Vivaldi, Queen Mary, University of London

Introduction

Exercise 1: Possible description

- --

Introduction

- Although we are not going to learn real analysis on our maths course, we need to learn to communicate well at the level we are at.

- Learning to communicate mathematics helps us to clarify our thinking and understanding about what we are presenting.

Introduction

Example 4: A square is …

1) … a geometric object consisting of four sides;

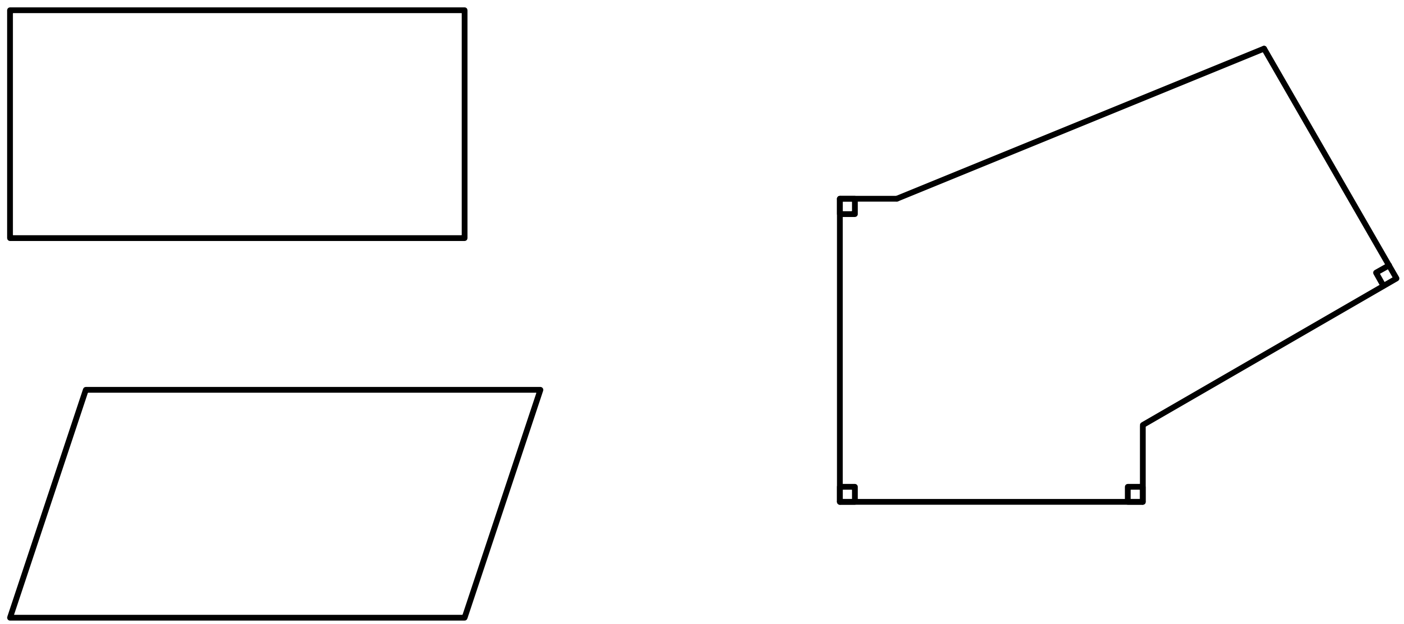
2) … a polygon with four 90^0 angles;

3) … a polygon with four equal sides and four 90^0 angles;

Introduction

Example 4:

- None of the definitions are correct.



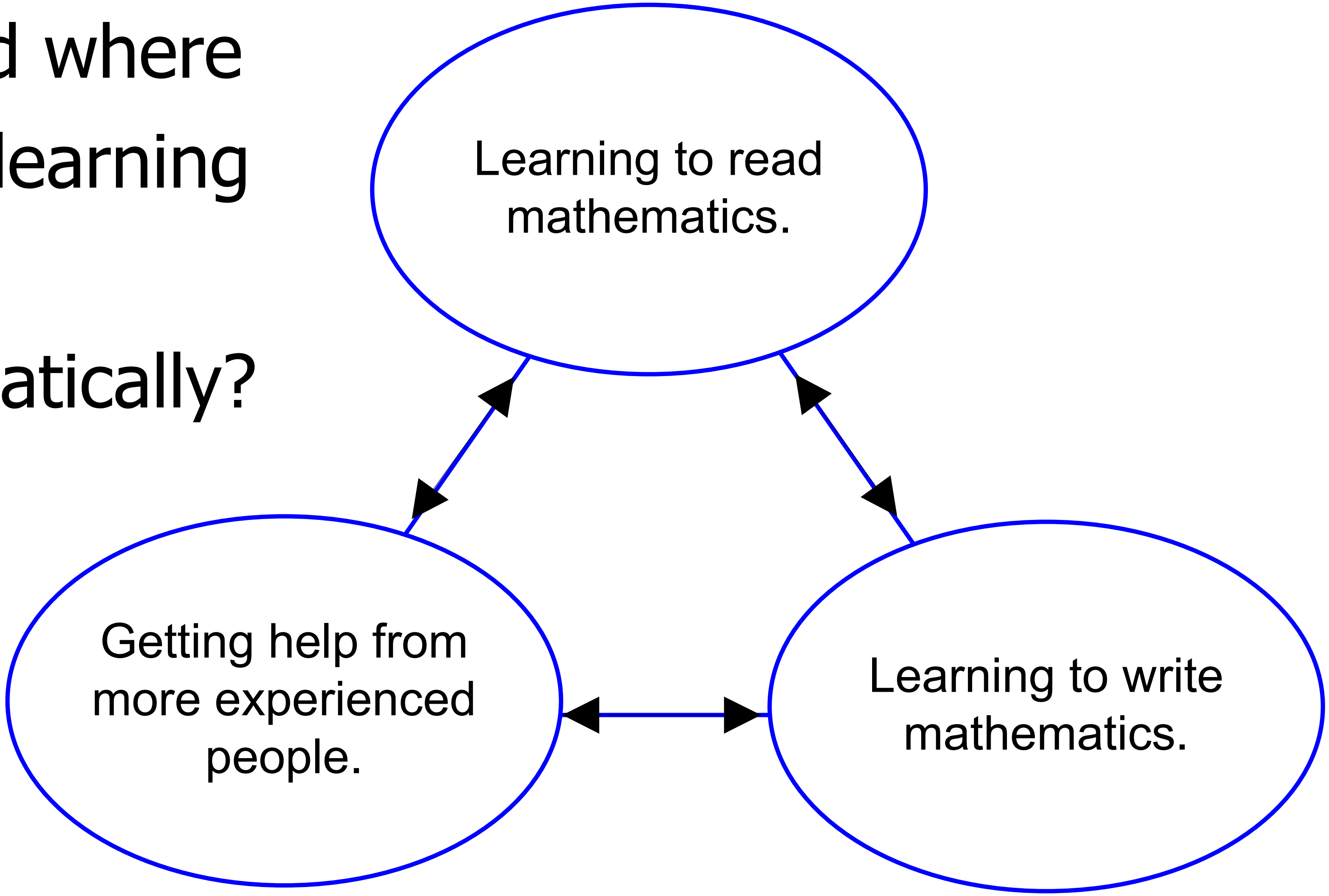
Introduction

- Learning to communicate mathematics helps us identify weaknesses or misunderstandings we may have.
  - This is good. Why?

- Learning to communicate mathematics highlights the unexpected depth and precision required for doing mathematics.

Introduction

How and where to start learning to write mathematically?



Introduction

- How and where to start:
  - Learning to read mathematics: Student journals, undergraduate textbooks.
  - Learning to write: Write a set of notes as if you were writing a chapter of a textbook for another person: novice or expert;
  - Getting help: Find a mathematician and/or maths teacher to review your writing.

Example

- Some of the ideas below come from Franco Vivaldi at Queen Mary, London.
- The following is complex in terms of the mechanics of differentiation …

Find the derivative of the following function

$$f(x) = \frac{\log(2 + \sin x^2)}{e^x}.$$

Introduction

Example 5

- .. but needs limited conceptual understanding.

Find the derivative of the following function

$$f(x) = \frac{\log(2 + \sin x^2)}{e^x}.$$

Introduction

Example 5

- We can do the task without understanding the mathematical meaning of the words maths terms.

Find the derivative of the following function

$$f(x) = \frac{\log(2 + \sin x^2)}{e^x}.$$

Introduction

Example 5

- But, what is "function", "derivative", "log", "sin", "e^x"? How/why are they defined?

Find the derivative of the following function

$$f(x) = \frac{\log(2 + \sin x^2)}{e^x}.$$

Introduction

- A dictionary is constantly needed …
- … i.e. you need to clearly understand the mathematical meaning of terms so that you can use them correctly in your communication
  - "A function is defined to be …";
  - "The derivative of a function is defined to be …";
  - "log(x) is defined to be …".

Introduction

- We all need explicit teaching on how to read and write mathematics.

- **Example 6**
  a) A continuous function can be represented by a curve.

  b) A continuous function can be *illustrated (on a graph)* by a curve

Introduction

- We all need explicit teaching on how to read and write mathematics.

- **Example 6**

  a) A continuous function can be represented by a curve

  b) A continuous function can be represented by a *power series.*

Introduction

- We all need explicit teaching on how to read and write mathematics.

- **Example 6**
  a) A continuous function can be represented by a curve.

  b) A continuous function can be *illustrated* by a curve which is a geometric *representation*.

Introduction

Function ←represented as→ Power series

(analytic) (analytic)

Function ←illustrated as→ Curve

(analytic) (geometric)

Introduction

Curve ⟷ (represented as) Infinite number of infinitely small straight lines

(geometric) (geometric)

Introduction

- **Example 7**

  a) The derivative of a function at a point x = a is the value of the limit at that point

  b) The derivative of a function at a point x = a is the value of the limit *of the difference quotient* at that point.

Density of mathematics itself

- **Examples**

Consider the following mathematical statement

Given $f : \mathbb{R} \to \mathbb{R}$ and f is $C^1(a, b)$, $\exists\, c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}\ .$$

versus …

Density of mathematics itself

- **Examples**

Consider the following mathematical statement

Given a continuous function $f(x)$ defined over the interval $[a, b]$, and having a continuous first derivative in the interval (a, b), then there exists a value c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \; .$$

Density of mathematics itself

- It is easier to read a sentence composed of a mixture of mathematical symbolism and mathematical English than just mathematical symbolism.

Differentiation phrasing

Exercise 2

- Transform the following statement into a correct conceptual description containing as much mathematical English as possible.

Let f be continuous on $[a, b]$ and let $f(a) < 0 < f(b)$.

Then $\exists c \in [a, b]$ such that $f(c) = 0$.

Differentiation phrasing

Exercise 2

- ***Draft 1***

Let us have a real valued function which is continuous on a closed interval given by a lower bound a and an upper bound b, and let the function evaluated at a be negative, and let the function evaluated at b be positive. (*let there be a change of sign when ...*)

Density of mathematical sentences

- **Watch out for dense mathematical-English sentences:** When writing mathematical English sentences it is helpful to not make their construction too "dense".

- **Example 8**

"Let f(x) be a continuous,

infinitely differentiable invertible function".

Density of mathematical sentences

- Since each word has a density of mathematical meaning, this sentence requires some effort to read.
- How can we change the previous sentence to make it easier to read?

“Let f(x) be a continuous function
which is infinitely differentiable
and for which there exists an inverse”,

Collocation of mathematical terminology

Exercise 4

- Make the following sentences more readable:
  - The first principles definition of the derivative implies equality of left and right limits of the difference quotient.
  - An infinitely differentiable continuously bounded functions defined on a real open interval.

Collocation of mathematical terminology

Exercise 4

- An infinitely differentiable continuously bounded function defined on a real open interval.

- A continuously-bounded function is infinitely differentiable on a real open interval: **NO**

- An infinitely differentiable function which is continuous, and bounded, is defined on an open interval.

Density of mathematics itself

Exercise 5

- Consider the following symbols
  - ε and δ are positive numbers which are arbitrarily small;
  - $\exists$ means "there exists";
  - c is a real number;
  - $|\ |$ is the standard symbol for "modulus" or "absolute value".

Density of mathematics itself

Exercise 5

- Make the following mathematical statement more readable:

$$\text{A function } f: \mathbb{R} \to \mathbb{R} \text{ is continuous at } c \in \mathbb{R} \text{ if,}$$

$$\text{given } \epsilon > 0, \exists \delta > 0 \text{ such that}$$

$$|x - c| < \delta \Longrightarrow |f(x) - f(c)| < \epsilon.$$

Collocation of mathematical terminology

- Standard collocations and phrases: these are mathematical English words which are usually always seen together in pairs or as a group:
  - Two-sided limit;
  - Limit from the left / Limit from the right;
  - First principles;
  - Rate-of-change;
  - Instantaneous rate-of-change;

Collocation of mathematical terminology

- Standard collocations and phrases:
  - Average rate-of-change;
  - Derivative at a point;
  - Function $f(x)$ is differentiable at $x = c$ when ...
  - Global maximum or minimum;
  - Local maximum or minimum;
  - Local or global extrema;

Collocation of mathematical terminology

- Standard collocations and phrases:
  - The derivative of a sum;
  - The sum of the derivatives;
  - An increasing and bounded function;
  - The product rule for differentiation;
  - The quotient rule for differentiation;
  - Implicit differentiation;

Collocation of mathematical terminology

- Standard collocations and phrases:
  - A function defined over/on an interval;
  - An open interval;
  - A closed interval;
  - A semi-open or semi-closed interval;

================

Answers to exercises

================

Introduction

Exercise 1: Possible description

This is a smooth function, which is bounded and non-negative. It features an infinite sequence of evenly spaced local maxima, whose height decreases monotonically to zero. The function has a zero between any two consecutive maxima.

From p87, “Mathematical Writing for undergraduate students”, Franco Vivaldi, Queen Mary, University of London

Differentiation phrasing

Exercise 2

- Possible description:

Let f be a continuous function defined on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

Now let the function vanish when evaluated at $x = a$ and $x = b$. Then there exists a real number c in the open interval such that the derivative vanishes at c (or such that there is a stationary point at c).

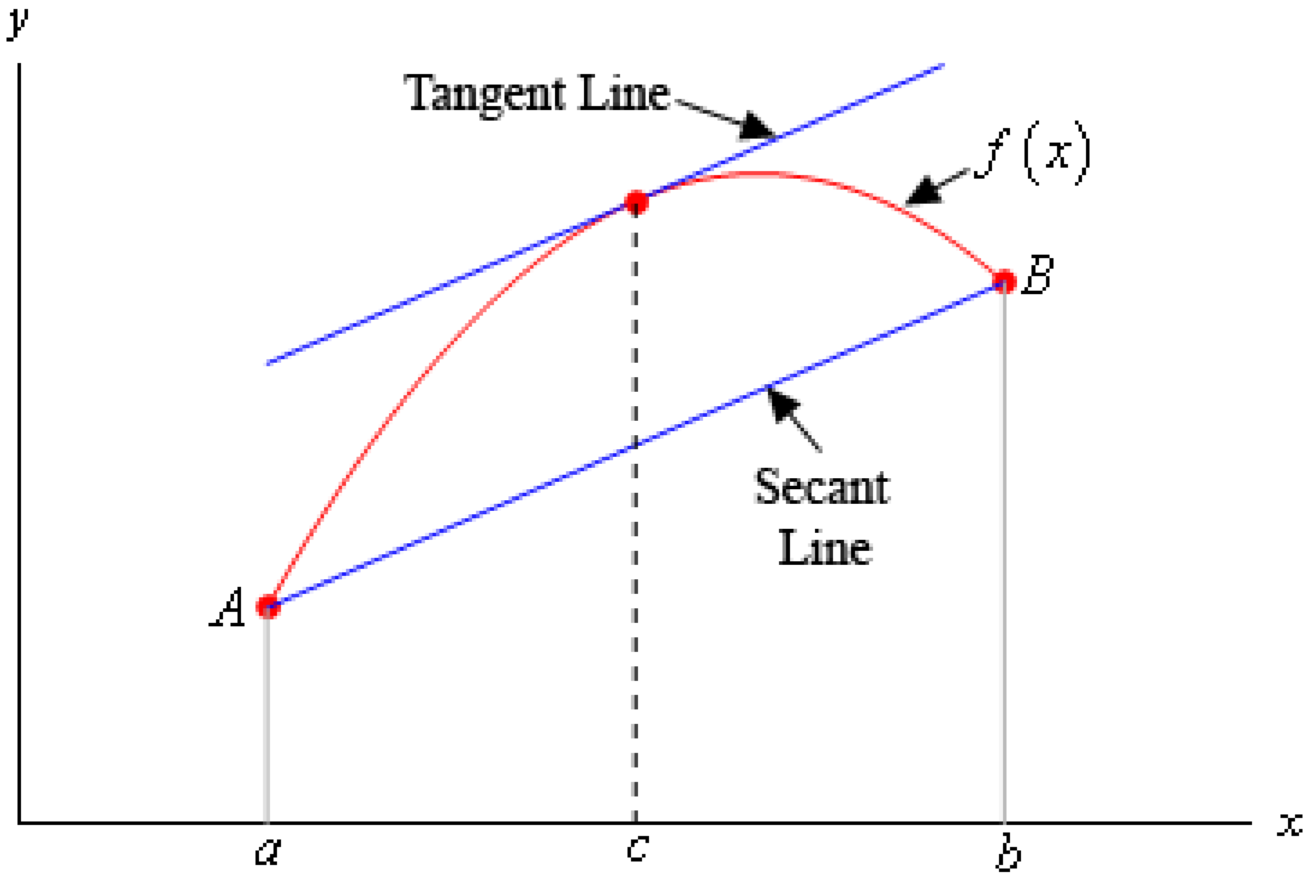
================

Appendix

================

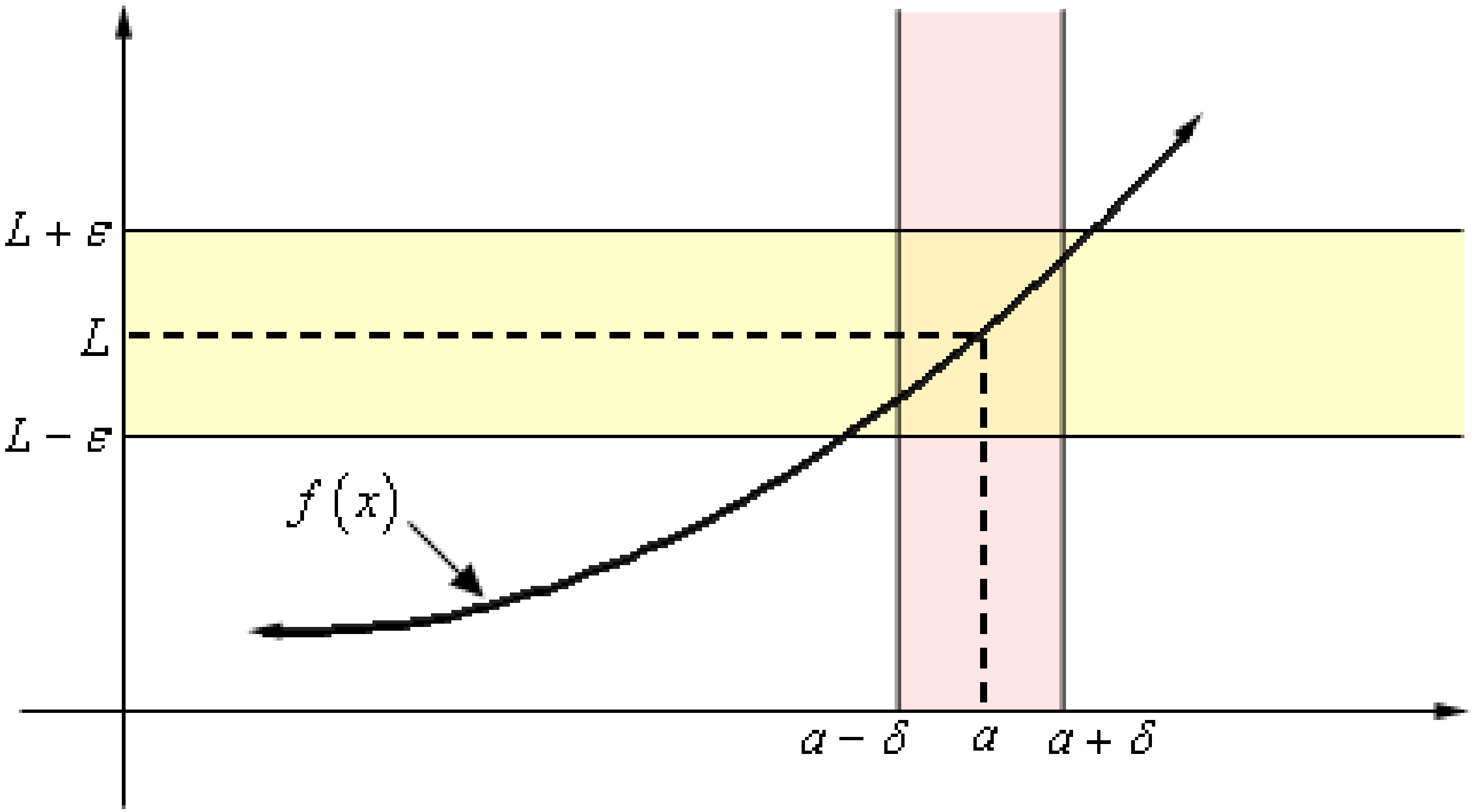
Differentiation terminology

- What are the basic terms of differentiation?



Differentiation terminology

- What are the basic terms of differentiation?



Introduction

Example 3

1) A function f tends to a limit $L \in \mathbb{R}$ as x tends to $c \in \mathbb{R}$ if,

for all $\epsilon > 0$, then there exists $\delta > 0$ such that

$|x - c| < \delta$ implies $|f(x) - L| < \epsilon$.

2) We can make $f(x)$ arbitrarily close to L

provided that we take x sufficiently close to c

Example

- ===

Essential dictionary: from symbols to words

$$(1-x, 1+x^2, 1-x^3, \ldots, 1+(-x)^n, \ldots)$$

- A sequence.
- An infinite sequence.
- An infinite sequence of polynomials.
- An infinite sequence of polynomials in one indeterminate.

put symbols in a context:

with integer coefficients.

with increasing degree.

with bounded coefficients.

. . .

Example

Do as here with diff phrasing

Essential dictionary: from symbols to words

$$(1-x, 1+x^2, 1-x^3, \ldots, 1+(-x)^n, \ldots)$$

- A sequence.
- An infinite sequence.
- An infinite sequence of polynomials.
- An infinite sequence of polynomials in one indeterminate.

put symbols in a context:

with integer coefficients.
with increasing degree.
with bounded coefficients.
. . .

Differentiation phrasing

Exercise *: Do as here with diff*

Exercise 4. The following expressions define sets. Turn words into symbols.

1. The set of negative odd integers.
2. The set of natural numbers with three decimal digits.
3. The set of rational numbers which are the ratio of odd integers.
4. The set of rational numbers between 3 and π.
5. The set of real numbers at distance 1/4 from an integer.
6. The complement of the unit circle in the Cartesian plane.
7. The set of lines tangent to the unit circle.

$$\{n \in \mathbb{N} \,:\, 10^2 \leqslant n < 10^3\}$$

Differentiation phrasing

Exercise

Do as here with diff phrasing

Exercise 5. For each expression, provide two levels of description: [$\not{e}$]

i) a coarse description, which only identifies the object's type (set, function, equation, statement, etc.);
ii) a finer description, which defines the object in question or characterises its structure.

1. $x^3 - x - 2$ polynomial
2. $x^3 - x - 2 = 0$ equation
3. $3^3 + 4^3 + 5^3 = 6^3$ identity
4. $x - y > 0$ inequality
5. $x = x + 1$
6. $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
7. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
8. $2\mathbb{Z} \supset 4\mathbb{Z}$ sentence
9. $(\mathbb{Q} \setminus \mathbb{Z})^2$ set
10. $(a_1, a_3, a_5, \ldots)$ sequence
11. $((x_1), (x_1, x_2), (x_1, x_2, x_3), \ldots)$
12. $\sin \circ \cos$ function

Differentiation phrasing

Exercise *: Do as here with diff*

Exercise 8. Let $f:\mathbb{R}\to\mathbb{R}$. Rewrite each symbolic sentence without symbols, apart from f.

1.	$f(0)\in\mathbb{Q}$	2.	$f(\mathbb{R})=\mathbb{R}$
3.	$\#f(\mathbb{R})=1$	4.	$f(\mathbb{Z})=\{0\}$
5.	$0\in f(\mathbb{Z})$	6.	$f^{-1}(\{0\})=\mathbb{Z}$
7.	$f(\mathbb{R})\subset\mathbb{Q}$	8.	$f(\mathbb{R})\supset\mathbb{Z}$
9.	$f(\mathbb{Z})=f(\mathbb{N})$	10.	$f(\mathbb{Q})\cap\mathbb{Q}=\emptyset$
11.	$f^{-1}(\mathbb{Q})=\emptyset$	12.	$\#f^{-1}(\mathbb{Z})<\infty.$

The image of the set of integers under the function f is the set consisting of the integer 0. [Robotic, no understanding]

The function f vanishes at all integers. [Good]

General linking vocabulary

- Linking terms or phrase:

Hence	Therefore	So
Implies	Simplifying (we get)	Factorising (we obtain)
Dividing by … (we get)	Multiplying both sides by …	Comparing left and right hand sides
Substituting … we get …	Given that …	We see that …

General linking vocabulary

- Linking terms or phrase:

For all …	There exists …	Such that …
The value …	Satisfies …	The exact value of …
The approximate value of … to 2 decimal places	Because (of) …	Since …
We have …	It follows that …	Let …

General linking vocabulary

- Linking terms or phrase:

Hence …	Implying / This implies that …	

Collocation of mathematical terminology

- Note that there are mathematical words that usually (always) go together:
  - "Let $f(x)$ be a continuous function", **Yes**
  - "Let $f(x)$ be a function which is continuous"; **No**

Collocation of mathematical terminology

- Note that there are mathematical words that usually (always) go together:

  – "Let $f(x)$ be a function which is infinitely differentiable"; **Yes**

  – "Let $f(x)$ be a function which is differentiable an infinite number of times"; **No**